ADVANCES IN MATHEMATICS 13, 274–283 (1974)

Bifurcations Under Nongeneric Conditions

N. KOPELL

Mathematics Department, Northeastern University, Boston, Massachusetts 02115

AND

L. N. HOWARD

Mathematics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Much interest has been focused recently on bifurcation theorems (cf. [1] for bibliography), which state, for example, that if the linearization around a critical point of a differential equation satisfies some conditions, then the (nonlinear) equation has some special solutions near the critical point, e.g., other critical points or periodic solutions. A well known bifurcation theorem (e.g., [1, 2]), can be stated, in its simplest form, as follows:

THEOREM 1. *Let $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$ be a one-parameter family of autonomous differential equations on R^n, $\mathbf{F}$ depending smoothly (e.g., C^2) on all of its $n + 1$ arguments, such that $\mathbf{X} = 0$ is a critical point for each μ with $|\mu|$ sufficiently small (i.e., $\mathbf{F}(0, \mu) = 0$). Let A_μ be the linearization (with respect to $\mathbf{X}$) around $\mathbf{X} = 0$ of $\mathbf{F}(\mathbf{X}, \mu)$. Suppose that for $\mu = 0$, $\lambda = 0$ is a simple eigenvalue of A_μ and let $\lambda(\mu)$ be the eigenvalue of A_μ which reduces to $\lambda = 0$ for $\mu = 0$. Suppose further that $(d\lambda/d\mu)(0) \neq 0$.*

Then in every sufficiently small neighborhood of $(0, 0)$ there is a curve in the $n + 1$ dimensional $(\mathbf{X}, \mu)$ space passing through $\mathbf{X} = 0$, $\mu = 0$, and distinct from the μ axis, each of whose $\mathbf{X}$ components is a critical point for $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$. There are no other critical points in a neighborhood of $\mathbf{X} = 0$, $\mu = 0$.

This particular bifurcation theorem really just concerns the zeroes of a family of vector valued-functions. However, its main interest is in connection with solutions to differential equations. This form of the theorem was stated by Hopf [2], and is parallel to the "Hopf bifurcation theorem" [1, 2, 9] which concerns periodic orbits arising when a pair of simple eigenvalues transversely cross the imaginary axis.

Ruelle [4] proves a generalization of this theorem to equations whose linearization is equivariant under a group action. He uses as a hypothesis the same nondegeneracy conditions as above, e.g., that $\lambda = 0$ is a *simple* eigenvalue of A_0. In some cases, however, symmetry requirements rule out that condition. For example, the critical points of Hamiltonian vector fields have eigenvalues which occur conjugate about the real axis as well as the imaginary axis [5]. There is no Hamiltonian vector field which has a simple eigenvalue $\lambda = 0$; for a one-parameter family $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$ of Hamiltonian vector fields, eigenvalues must pass through $\lambda = 0$ in pairs. This illustrates the fact that what is non-generic in the space of all vector fields may be generic for those satisfying some symmetry conditions.

The purpose of this paper is to prove the conclusion of Theorem 1 under hypotheses that are considerable weaker. In particular, we allow A_0 to have $\lambda = 0$ as a k-fold eigenvalue. The nondegeneracy condition $(d\lambda/d\mu)(0) \neq 0$ is replaced by another condition which is equivalent if $k = 1$, namely: $(d/d\mu)[\det A_\mu](0) \neq 0$. The result applies to Hamiltonian systems, but is more general. It also applies where there is no underlying symmetry of the operator. However, it is shown here that the condition $(d/d\mu)(\det A_\mu)(0) \neq 0$ implies some symmetry of the spectrum: the k branches of the eigenvalues of the A_μ which pass through $\lambda = 0$ are asymptotically like the roots of $\lambda^k - \mu = 0$. We also discuss the behavior of the differential equation near the new critical points.

For other related bifurcation theorems, see [3, 6] and [7].

THEOREM 2. *Let $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$ be a one-parameter family of autonomous differential equations on R^n such that F depends C^2 smoothly on its $n + 1$ arguments and $\mathbf{F}(0, \mu) = 0$ for $|\mu|$ sufficiently small. Let A_μ be as defined in Theorem* 1. *Let $D(\mu) = \det A_\mu$. Assume that $D(0) = 0$, and*

$$\frac{d}{d\mu} D(0) \neq 0. \tag{1}$$

Then the conclusion of Theorem 1 *holds.*

Proof. Since $D(0) = 0$, there is a vector $\mathbf{v}$ such that $A_0(\mathbf{v}) = 0$;

we may assume that $\mathbf{v} = (1, 0, \ldots, 0)$. We first show that the hypothesis $(d/d\mu)\, D(0) \neq 0$ implies that for some I, $1 \leqslant I \leqslant n$,

$$\frac{\partial(F_1, \ldots, \hat{F}_I, \ldots, F_n)}{\partial(x_2, \ldots, x_n)}(0, 0) \neq 0. \tag{2}$$

where $\hat{F}_I$ denotes the omission of F_I, and

$$\frac{\partial^2 F_I}{\partial x_1\, \partial\mu}(0, 0) \neq 0, \tag{3}$$

where $\mathbf{F} = (F_1, \ldots, F_n)$ and $F_j = F_j(\mathbf{X}, \mu)$. This is true because of the following: $D(\mu) = (\partial(F_1, \ldots, F_n)/\partial(x_1, \ldots, x_n))(0, \mu)$. If we use the formula for differentiating a determinant by columns, we find that, with the above choice of coordinates, the only (possibly) nonzero terms are those coming from differentiating the first column, i.e., those of the form $(\partial^2 F_I/\partial x_1\, \partial\mu)(0, 0)(\partial(F_1, \ldots, \hat{F}_I, \ldots, F_n)/\partial(x_2, \ldots, x_n))(0, 0)$. (All other terms involve a determinant whose first column is identically zero.) Hence (1) implies (2) and (3).

Now (2) implies that the $(n - 1)$ equations $F_j(\mathbf{X}, \mu) = 0$, $j \neq I$, may be solved for $x_2, \ldots, x_n$ in terms of x_1 and μ (for $\mathbf{X}$, μ small), i.e., $x_j = h_j(x_1, \mu)$. Also, $h_j(x_1, \mu)$ has the form

$$h_j(x_1, \mu) = x_1{}^2 g_j(x_1, \mu) + \mu x_1 \tilde{g}_j(x_1, \mu)$$

where g_j and $\tilde{g}_j$ are continuous. (For the linear terms in x_1 are missing from the equations $F_j(\mathbf{X}, \mu) = 0$, except where multiplied by μ.) Hence

$$\frac{\partial h_j}{\partial x_1}(0, 0) = 0. \tag{4}$$

It remains to solve the equation $F_I(\mathbf{X}, \mu) = 0$ for a curve of critical points. We wish to get a curve other than $(\mathbf{X}, \mu) = (0, \mu)$. We first notice that the equation $F_I(\mathbf{X}, \mu) = 0$ is divisible by x_1. For $F_I(0, \mu) = 0$, so $F_I = \sum_{j=1}^{n} x_j f_j(\mathbf{X}, \mu)$. Since x_j is divisible by x_1, this shows that F_I has the form $x_1 g(x_1, \mu)$.

The curve $x_1 g(x_1, \mu) = 0$ has two branches: $x_1 = 0$, which is the curve $(x_1, \mu) = (0, \mu)$, and $g(x_1, \mu) = 0$. To solve $g(x_1, \mu)$ for a unique curve $\mu = \bar{\mu}(x_1)$ (near $x_1 = 0$) it is sufficient that $(\partial g/\partial\mu)(0, 0) \neq 0$. But

$$\begin{aligned}(\partial g/\partial\mu)(0, 0) &= (\partial^2 F_I/\partial\mu\, \partial x_1)(0, 0) + \sum_{j\neq 1} (\partial^2 F_I/\partial\mu\, \partial x_j)(0, 0)(\partial h_j/\partial x_1)(0, 0) \\ &= (\partial^2 F_I/\partial\mu\, \partial x_1)(0, 0) \neq 0\end{aligned}$$

by (3) and (4).

REMARKS

1. Hypothesis (1) implies that A_0 has rank $n-1$. (Otherwise all $(n-1)$ dimensional minors of A_0 would vanish.) This condition means that the linear system $\mathbf{X}_t = A_0\mathbf{X}$ has exactly one line of critical points. It has been pointed out to us by P. Rabinowicz that Theorem 2 is equivalent to the finite dimensional case of Theorem 1.7 of [3]. In our terminology, the hypotheses of 1.7 would be that A_0 has rank $n-1$ and $((d/d\mu)\, A_\mu\,|_{\mu=0})\mathbf{v}$ not in range A_0. The proof that these hypotheses are equivalent to $(dD/d\mu)(0) \neq 0$ makes use of arguments somewhat similar to those in the proof given above.

2. The index of the critical point $\mathbf{X} = 0$ of $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$ is $\operatorname{sgn}\det A_\mu$, if $\det A_\mu \neq 0$. Hence $(d/d\mu)\, D(0) \neq 0$ implies that the critical point $\mathbf{X} = 0$ changes its index as μ changes sign; this implies, by topological methods, that there must be new critical points in a neighborhood of $(\mathbf{X}, \mu) = (0, 0)$ [6, 8]. The transversality condition (1) implies that there is at most one other curve of critical points.

3. First order systems of equations which come from a single higher order equation may satisfy the hypothesis of Theorem 2 without satisfying that of Theorem 1. Consider the 1-parameter family of nth order equations

$$y^{(n)} + a_{n-1}(\mu)\, y^{(n-1)} + \cdots + a_0(\mu)\, y + N(y, y', \ldots, y^{(n-1)}, \mu) = 0,$$

where $a_i(\mu)$ and $N(y, y', \ldots, y^{(n-1)}, \mu)$ are smooth and $N = o(\sum |\, y^{(i)}\,|)$. The equation can be made into a system in the usual way: let $x_1 = y$ and $x_{i+1} = x_i'$ for $1 \leqslant i \leqslant n-1$. Then

$$x_n' = -a_{n-1}(\mu)\, x_{n-1} - \cdots - a_0(\mu)\, x_1 - N(x_1, \ldots, x_n, \mu).$$

If $N(0, \mu) = 0$ for all μ, $x_1 = \cdots = x_n = 0$ is a critical point for all μ. The system satisfies (1) if $a_0'(0) \neq 0$. Note that if $a_i(0) = 0\ \forall i$, A_0 has an n-fold eigenvalue, so the system does not satisfy the hypothesis of Theorem 1.

4. The hypothesis of Theorem 2 is satisfied by many Hamiltonian systems. The simplest example has Hamiltonian

$$H(x_1, x_2) = (x_1^2/2) - (\mu/2)\, x_2^2.$$

Then $\dot{x}_1 = -(\partial H/\partial x_2) = \mu x_2$, $\dot{x}_2 = (\partial H/\partial x_1) = x_1$.

5. Note that the multiplicity k of the zero eigenvalue in Theorem 2 may be any integer k. This contrasts with the phenomena considered by Rabinowitz [6, Theorem 1.3] in which k odd was needed.

Related Results

1. Further conclusions can be drawn about the nature of the new critical points whose existence is asserted by Theorems 1 and 2. We denote these new critical points by $\bar{\mathbf{X}}(x_1)$, $\bar{\mu}(x_1)$, $(\bar{x}_1 = x_1)$. Under the (simple eigenvalue) hypothesis of Theorem 1, the following was essentially reported by Hopf [2]: Let $\lambda(\mu)$ be the (real) eigenvalue of A_μ which satisfies $\lambda(0) = 0$. There is an eigenvalue $\Lambda(x_1)$ at the critical point $\bar{\mathbf{X}}(x_1)$, $\bar{\mu}(x_1)$, such that $\Lambda(x_1) \to 0$ as $x_1 \to 0$. Then

$$\lim_{x_1 \to 0} \left[\frac{\Lambda(x_1)}{\lambda(\bar{\mu}(x_1))} + \frac{x_1}{\bar{\mu}} \frac{d\bar{\mu}}{dx_1}\right] = 0.$$

Hopf actually dealt only with the cases

$$\bar{\mu} = \kappa x_1 + 0(x_1^2) \qquad \text{or} \quad \bar{\mu} = \kappa x_1^2 + 0(x_1^3).$$

He stated the result as: $(d\Lambda/dx_1)(0) = -(d\bar{\mu}/dx_1)(0)(d\lambda/d\mu)(0)$ if $\bar{\mu} \sim \kappa x_1$ and $(d\Lambda/dx_1)(0) = -(2d\bar{\mu}/dx_1)(0)(d\lambda/d\mu)(0)$ if $\bar{\mu} \sim \kappa x_1^2$. The other eigenvalues at $\bar{\mathbf{X}}(x_1)$, $\bar{\mu}(x_1)$ are close to the remaining eigenvalues at $\mathbf{X} = 0$, $\mu = \bar{\mu}(x_1)$.

From the above some facts about stability of the critical points $\bar{\mathbf{X}}(x_1)$, $\bar{\mu}(x_1)$ emerge: If $\bar{\mu}(x_1) = \kappa x_1^p + 0(x_1^{p+1})$, for any $p > 0$, then $(x_1/\bar{\mu})(d\bar{\mu}/dx) \to -p$. Hence the eigenvalues $\Lambda(x_1)$ and $\lambda(\bar{\mu}(x_1))$ have opposite signs, so if $\bar{\mathbf{X}} = 0$, $\mu = \bar{\mu}(x_1)$ is a stable critical point (all eigenvalues in the left half-plane), then $\bar{\mathbf{X}}(x_1)$, $\bar{\mu}(x_1)$ is not. If $\bar{\mathbf{X}} = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, this is often expressed by saying that the new critical points are stable (respectively unstable) if the bifurcation is "supercritical," i.e., $\mu > 0$ (respectively "subcritical," i.e., $\mu < 0$).

Under the weaker hypotheses of Theorem 2, there is an analog to the above results:

PROPOSITION. *Let $D(\mu)$ be the determinant of the linearization at $\bar{\mathbf{X}} = 0$, with $D(0) = 0$ and $D'(0) \neq 0$ as in Theorem 2. Let the lineariza-*

tion at the other family of critical points $(\bar{\mathbf{X}}(x_1), \bar{\mu}(x_1))$ *have determinant* $\bar{D}(x_1)$. *Then*

$$\lim_{x_1 \to 0} \left[\frac{\bar{D}(x_1)}{D(\bar{\mu}(x_1))} + \frac{x_1}{\bar{\mu}} \frac{d\bar{\mu}}{dx_1}\right] = 0. \tag{5}$$

Proof. We use the same notations as in the proof of Theorem 2, and recall from that proof that $\bar{\mu} = 0(x_1)$ and $\bar{x}_j = 0(x_1^2)$ for $j > 1$. Also, as noticed there,

$$D'(0) = \begin{vmatrix} \dfrac{\partial^2 F_1}{\partial x_1\, \partial \mu} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial^2 F_n}{\partial x_1\, \partial \mu} & & \cdots & \dfrac{\partial F_n}{\partial x_n} \end{vmatrix} \qquad (\text{at } \mathbf{X} = 0, \mu = 0). \tag{6}$$

Let $\bar{A} = [(\partial F_j(\bar{\mathbf{X}}, \bar{\mu})/\partial x_k)]$ be the matrix of the linearization at the critical point $\bar{\mathbf{X}}, \bar{\mu}$. Then differentiating the relations $F_j(\bar{\mathbf{X}}, \bar{\mu}) = 0$ with respect to x_1 gives:

$$\bar{A} \frac{d\bar{\mathbf{X}}}{dx_1} + \frac{\partial \mathbf{F}}{\partial \mu} \frac{d\bar{\mu}}{dx_1} = 0. \tag{7}$$

We now multiply (7) by the classical adjoint of $\bar{A}$, and look at the first component of the resulting vector equation. (This amounts to solving (7) for $(d\bar{x}_1/dx_1)$ $(= 1)$ by Cramer's rule, except that we do not divide by the coefficient determinant.) This is, since $(\operatorname{adj} \bar{A})\bar{A} = \bar{D}I$,

$$\bar{D}(x_1) + \frac{d\bar{\mu}}{dx_1} \left[\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(\mu, x_2, \ldots, x_n)}\right]_{\bar{\mathbf{X}}, \bar{\mu}} = 0. \tag{8}$$

(The Jacobian here, if expanded by its first column, is evidently equal to the product of the first row of adj $\bar{A}$ by the column $(\partial \mathbf{F}/\partial \mu)$.)

Now, $(\partial F_j/\partial \mu)(\bar{\mathbf{X}}, \bar{\mu}) = x_1(\partial^2 F_j/\partial x_1\, \partial \mu)(0, 0) + 0(x_1^2)$ since

$$(\partial F_j/\partial \mu)(0, \bar{\mu}) = 0$$

and all $\bar{x}_j$ for $j > 1$ are $0(x_1^2)$. Thus, recalling (6), we see that

$$\left[\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(\mu, x_2, \ldots, x_n)}\right]_{\bar{\mathbf{X}}, \bar{\mu}} = x_1 D'(0) + 0(x_1^2).$$

Dividing (8) by $D(\bar{\mu}(x_1)) = \bar{\mu}(x_1)[D'(0) + 0(x_1)]$ and letting $x_1 \to 0$ we obtain the asserted result (5).

This last result does not say as much about stability as the corresponding result for the simple eigenvalue case. If n is even and the determinant of the linearization at a new critical point is negative, then at least one eigenvalue is real and positive, so the critical point is unstable. Similarly, one gets instability if n is odd and the determinant is positive. In the other two cases, neither stability nor instability can be ruled out.

2. The requirement that $D(0) = 0$ and $(d/d\mu)\, D(0) \neq 0$ forces the eigenvalues of A_μ to have some symmetry. Assume for definiteness that A_0 has an n-fold zero eigenvalue. (Otherwise we should restrict our attention to those branches of the eigenvalues of A_μ which go through $\lambda = 0$.) The simplest such system has A_μ given by

$$A = \begin{pmatrix} 0 & & & \mu \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix},$$

with characteristic polynomial $P_\mu(\lambda) = \lambda^n - \mu$. The eigenvalues are the nth roots of μ. For $\mu > 0$ they are of the form $\lambda = \mu^{1\backslash n} \exp(2j\pi i/n)$, where $i^2 = -1$; for $\mu < 0$ they have the form

$$\lambda = |\, \mu \,|^{1/n} \exp((2j + 1)\pi i/n).$$

For $n = 2$, as μ passes from negative to positive the eigenvalues cross from pure imaginary to real and opposite.

The following proposition says that the condition (1) forces the eigenvalues of A_μ to behave very similarly to those of the above simple examples.

PROPOSITION. *Let A_μ be a one-parameter family of $n \times n$ matrices whose elements are continuously differentiable functions of μ. Let $a_0(\mu) = (-1)^n \det A_\mu$, and let $P_\mu(\lambda) = \lambda^n + a_{n-1}(\mu)\, \lambda^{n-1} + \cdots + a_0(\mu)$ be the characteristic polynomial of A_μ. Suppose that $\lambda = 0$ is an n-fold root of P_0, and that $C \equiv a_0'(0) \neq 0$. Then for any $\epsilon > 0$ there is a ρ such that whenever $0 < |\, \mu \,| < \rho$, for each of the n-th roots $\nu = (-C\mu)^{1/n}$ there is an eigenvalue of A_μ of the form $\lambda = \nu(1 + g)$ where $|\, g \,| < \epsilon$. As $\mu \to 0$ from above or below these eigenvalues approach the origin along curves which are smoothly parameterized by μ for $|\, \mu \,| > 0$.*

Proof. We consider first the polynomial equation

$$\theta^n - 1 = \beta_0 + \beta_1\theta + \cdots + \beta_{n-1}\theta^{n-1} \tag{9}$$

and note that for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that whenever all $|\beta_i| < \delta$, (9) has a root $\theta_1 = 1 + \phi$, where $|\phi| < \epsilon$.

Now, since the $a_i(\mu)$ are C^1 differentiable and zero at $\mu = 0$ and since $C \neq 0$, we can write $a_i(\mu) = \mu C b_i(\mu)$ where the b_i are continuous and $b_0(0) = 1$. Then given $\epsilon > 0$, choose ρ so that $|\mu| < \rho$ implies $|b_i(\mu)| \, |C\mu|^{i/n} < \delta(\epsilon)$ for $1 \leqslant i < n$ and $|b_0(\mu) - 1| < \delta(\epsilon)$.

For a fixed μ (positive or negative) satisfying $0 < |\mu| < \rho$ let ν be any one of the nth roots of $-C\mu$, and set $\lambda = \alpha\nu$ in the characteristic equation $P_\mu(\lambda) = 0$, which then becomes

$$(\alpha\nu)^n + C\mu(b_0(\mu) + \alpha\nu b_1 + \cdots + (\alpha\nu^{n-1}b_{n-1}) = 0$$

or, cancelling the factor $C\mu = -\nu^n$:

$$\alpha^n - 1 = b_0 - 1 + \alpha(\nu b_1) + \cdots + \alpha^{n-1}(\nu^{n-1}b_{n-1}). \tag{10}$$

Since $|\mu| < \rho$ all the coefficients on the right in (10) are smaller than δ in absolute value, and comparing with (9) we see that (10) has a root $\alpha_1 = 1 + g$ where $|g| < \epsilon$. This means that there is an eigenvalue as asserted for each of the n choices of ν. Assuming that ϵ is small enough, the different choices evidently must give distinct eigenvalues, so for small enough non-zero μ there are n distinct eigenvalues of the asserted form.

If μ is now allowed to vary and approach zero (from either the right or the left) we see that the eigenvalues must vary smoothly with μ, since simple roots of monic polynomials vary smoothly with the coefficients, which here depend smoothly on μ.

3. Condition (1) is the weakest condition that can be placed purely on the linear part of $\mathbf{F}(\mathbf{X}, \mu)$ (with respect to $\mathbf{X}$) and still have the conclusion of Theorem 2 hold (for all possible higher order terms). More precisely:

PROPOSITION. *Let A_μ be a one-parameter family of $n \times n$ matrices whose entries depend smoothly (C^2) on μ. Let $D(\mu) = \det A_\mu$. Suppose that $D(0) = 0$ and $(d/d\mu)\, D(0) = 0$. Then there are mappings $\mathbf{F}(\cdot, \mu)$: $R^n \to R^n$ such that $\mathbf{F}(0, \mu) = 0$, $(\partial \mathbf{F}/\partial \mathbf{X})(0, \mu) = A_\mu$, and the set of critical points in $(\mathbf{X}, \mu)$ space, near $\mathbf{X} = 0$, $\mu = 0$, consists of at least two curves other than $(0, \mu)$.*

Proof. Since $D(0) = 0$, we may assume, as in Theorem 2, that the first column of A_0 is identically zero. First suppose that the rank of A_0 is less than $n - 1$, so the null-space N of A_0 has at least two dimensions. Then we may let $\mathbf{F}(\mathbf{X}, \mu) = A_\mu \mathbf{X}$; the critical points of $\mathbf{X}_t = \mathbf{F}(\mathbf{X}, \mu)$ contain the two dimensional set $N \times \{0\}$ in $(\mathbf{X}, \mu)$ space. Hence we may assume that A_0 has rank $n - 1$.

This implies that $n - 1$ of the n equations $A_\mu \mathbf{X} = 0$ in the $n + 1$ unknowns $x_1, \ldots, x_n, \mu$ may be solved for $x_2, \ldots, x_n$ in terms of x_1 and μ (near $x_1 = \mu = 0$). Substituting for $x_2, \ldots, x_n$ in the unused equation (say it is the ith), this equation takes the form $a(\mu)x_1 = 0$, where $a(\mu) = 0(\mu^2)$. (This follows from arguments in the proof of Theorem 2, using the fact that $D(0) = 0$ and $D'(0) = 0$.)

To finish the proof and construct $\mathbf{F}(\mathbf{X}, \mu)$ we argue by cases. In all cases, $F_j(\mathbf{X}, \mu) = A_\mu{}^j \mathbf{X}$, $j \neq i$, where $A_\mu{}^j$ is the jth row of A_μ.

i. Suppose $\mu = 0$ is not an isolated zero of $a(\mu)$. Then for each μ_* satisfying $a(\mu_*) = 0$, and any x_1, $(x_1, x_2(x_1, \mu^*), \ldots, x_n(x_1, \mu_*), \mu)$ is a critical point for $\mathbf{X}_t = A_\mu \mathbf{X}$. Hence there are infinitely many curves of such critical points in any neighborhood of $\mathbf{X} = 0$, $\mu = 0$. Therefore we may let $F_i(\mathbf{X}, \mu) = A_\mu{}^i \mathbf{X}$.

ii. Suppose $\mu = 0$ is an isolated zero and $a(\mu) \geqslant 0$ (respectively $\leqslant 0$) for all $|\mu|$ sufficiently small. Let $F_i(\mathbf{X}, \mu) = A_\mu{}^i \mathbf{X} - x_1{}^3$ (respectively $A_\mu{}^i \mathbf{X} + x_1{}^3$). The zeroes of $\mathbf{F}(\mathbf{X}, \mu)$ satisfy $x_j = x_j(x_1, \mu)$, $j \neq 1$, and $A_\mu{}^i \mathbf{X} - x_1{}^3 = 0$, i.e., $a(\mu)x_1 - x_1{}^3 = 0$ (respectively $a(\mu)x_1 + x_1{}^3 = 0$). This last equation has two distinct curves of solutions, giving critical points different from the curve $\mathbf{X} = 0$. Note that if $F_i(\mathbf{X}, \mu) = A_\mu{}^i \mathbf{X} + x_1{}^3$ (respectively $A_\mu{}^i \mathbf{X} - x_1{}^3$), there are no critical points other than $\mathbf{X} = 0$.

iii. Suppose $\mu = 0$ is an isolated zero and $\operatorname{sgn} a(\mu) = \operatorname{sgn} \mu$ (respectively $-\operatorname{sgn} \mu$). Let $F_i(\mathbf{X}, \mu) = A_\mu{}^i \mathbf{X} - \mu x_1{}^3$ (respectively $A_\mu{}^i \mathbf{X} + \mu x_1{}^3$). Then, as before, the zeroes of $\mathbf{F}(\mathbf{X}, \mu)$ satisfy $x_j = x_j(x_1, \mu)$, $j \neq 1$ and $a(\mu)x_1 - \mu x_1{}^3 = 0$ (respectively $a(\mu)x_1 + \mu x_1{}^3$). The last equation is equivalent to $(a(\mu)/\mu) - x_1{}^2 = 0$ (respectively $a(\mu)/\mu + x_1{}^2 = 0$) and $a(\mu)/\mu \geqslant 0$ (respectively $\leqslant 0$). Hence, there are two distinct curves of solutions.

It remains an open question whether there is an analog to Theorem 2 concerning periodic orbits, i.e., is there a simple condition, analogous to (1), allowing multiple pairs of eigenvalues to cross the imaginary axis, (all at the same value of μ and all having the same imaginary values $\pm i\sigma$), which still yields the conclusion of the Hopf theorem: the existence of a one-parameter family of periodic solutions (with period near $2\pi/\sigma$)?

References

1. D. Sattinger, Topics in stability and bifurcation theory, Springer Lecture Notes #309, University of Minnesota, 1972.
2. E. Hopf, Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems, *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig* **94** (1942), 1.
3. M. Crandall and P. Rabinowitz, Bifurcation from simple eigenvalues, *J. Functional Analysis* **8** (1971), 321.
4. D. Ruelle, Bifurcations in the presence of a symmetry group, Institut des Hautes Études Scientifiques, 1972.
5. R. Abraham, "Foundations of Mechanics," Benjamin, New York, 1967.
6. P. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Functional Analysis* **7** (1971), 487.
7. K. Kirchgässner, Multiple Eigenvalue Bifurcation for Holomorphic Mappings, Contrib. to Nonlinear Functional Analysis, Academic Press, 1971.
8. D. Sattinger, Stability of bifurcating solutions by Leray–Schauder degree, *Arch. Rat. Mech. Anal.* **43** (1971), 154.
9. D. Ruelle and F. Takens, On the nature of turbulence, *Commun. Math. Phys.* **20** (1971), 167.